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ON THE MULTIPLICITY OF TENSORIAL FORMS FOR TENSOR CONNECTIONS ON DIFFERENTIABLE MANIFOLDS

Dr. Haresh Gambhir Chaudhari

Department Of Mathematics, Mgsm Dadasaheb Suresh G Patil College, Chopda.(India)

ABSTRACT

In this paper, we study the relational structures of tensor products over tangent spaces on a differentiable manifold X. Let β be a smooth curve parameterized on X, and let T_P,T_Q,T_R," and " T_S represent the tangent spaces evaluated at the sequential points P,Q,R," and " S along β. First, we establish conditions under which the tensor product spaces T_Q⊗T_P and T_S⊗T_R achieve structural equivalence. Second, utilizing linear isomorphism mappings that characterize parallel transport along the curve, we prove the existence of two distinct, valid tensorial forms associated with a singular tensor-connexion. These formulations provide extended mechanisms for tracking algebraic fields along smooth manifold paths.

Keyword: Differentiable manifold, Tangent spaces, Tensor-connexion, Tensorial forms, Parallel transport, Tensor product spaces, Multi-vectors

1. INTRODUCTION

In differential geometry, a connexion establishes a systematic framework for differentiating tensor fields along smooth curves on a differentiable manifold. By defining parallel transport maps between distinct tangent spaces, one can compare vectors and higher-order tensors across varying localized neighborhoods.Let X denote a differentiable manifold, and let β be a curve mapping through the points of X. For any two points P and Q situated along β, we denote the linear parallel transport map—which acts as an isomorphism from the tangent space T_P onto the tangent space T_Q—by the operator Ψ_{PQ}. The local representation of this connexion configuration at a parameterized point $x \in \beta$ gives rise to an associated tensorial form, denoted by Ψ_x. While classical treatments typically frame parallel transformations via individual connection forms, this paper explores the structural interplay between multiple interior points along a curve path. We first evaluate the containment and equivalence structures of tensor product spaces generated by four distinct tangent spaces (T_P, T_Q, T_R, T_S) bound by continuous interior connections. Following this, we investigate the formal existence of twin geometric tensorial forms (Ψ_x and ϕ_y) linked to a singular tensor-connexion operator f acting on multi-vector domains. These findings build upon foundational manifold structures outlined by Bishop, Crittenden, and Goldberg.

2. RESULTS AND DISCUSSION

2.1 Equivalence of Tangent Tensor Product Spaces:

Theorem 2.1: Let T_P, T_Q, T_R, and T_S be the tangent spaces evaluated at the points P, Q, R, and S, respectively, along the curve β on a differentiable manifold X. Then, for suitable spatial distributions, the tensor product space satisfies:

$$T_Q \otimes T_P = T_S \otimes T_R$$

Proof: Consider the parallel transport connexion Ψ_{QP} acting as a linear mapping from the tangent space T_Q onto the destination tangent space T_P, such that:

$$\Psi(T_Q) \subset T_P \quad (2.1)$$

Assuming that the points R and S occupy intermediate positions along the path of curve β between the endpoints Q and P, the composite mapping can be structurally decomposed into successive connections:

$$\Psi_{QP} = \Psi_{QR} \circ \Psi_{RS} \circ \Psi_{SP} \quad (2.2)$$

From the composition rule in (2.2) and the inclusion property in (2.1), it follows that the mappings satisfy the spatial containments:

$\Psi(T_R) \subset T_R$ and $\Psi(T_R) \subset T_S$ (2.3)with $\Psi(T_S) \subset T_P$ (2.4)

Taking the tensor product of these localized operator expressions reveals that:

$$\Psi(T_Q) \otimes \Psi(T_R) \otimes \Psi(T_S) \subset T_R \otimes T_S \otimes T_P \quad (2.5)$$

By applying the inclusion boundary (2.4) to a dual-space configuration, we can isolate the tracking parameters to show that:

$$\Psi(T_Q) \otimes T_P \subset T_Q \otimes \Psi(T_S) \quad (2.6)$$

Evaluating equations (2.5) and (2.6) simultaneously yields a direct subset relationship between the primary connection images:

$$\Psi(T_Q) \subset \Psi(T_S) \quad (2.7)$$

Because the connection mappings operate as linear isomorphisms along the smooth curve path, the operational inclusion implies direct containment of their underlying vector domains:

$$T_Q \subset T_S \quad (2.8)$$

Following an identical tracking sequence for the alternate interior points, we obtain the dual containment relation:

$$T_R \subset T_P \quad (2.9)$$

Combining the vector space containment conditions from (2.8) and (2.9), it becomes evident that for suitable choices of points $P, Q, R,$ and S along the manifold curve β, the tensor product spaces achieve algebraic equivalence:

$$T_Q \otimes T_P = T_S \otimes T_R$$

2.2 Duplicity of Tensorial Forms for a Tensor-Connexion

Theorem 2.2: There can exist two distinct tensorial forms associated with a singular tensor-connexion f.

Proof: Let f be a tensor-connexion. Let W_1^k denote the space of tensors of type $(k, 1)$ evaluated at the point $P \in \beta$, and let V_1^k denote the corresponding space of tensors of type $(k, 1)$ evaluated at the intermediate point $R \in \beta$. The action of the tensor-connexion maps these spaces such that:

$$f(W_1^k) \subset V_1^k \quad (2.10)$$

Let G represent the space of multi-vectors of arbitrary order at the point P, and let H represent the space of multi-vectors of arbitrary order at the point R. Following the mapping constraint established in (2.10), we restrict our focus to the space of tensors of type (0,2) at P (denoted as W_2) and the corresponding space of tensors of type (0,2) at R (denoted as V_2), which yields:

$$f(W_2) \subset V_2 \quad (2.11)$$

Next, we introduce two continuous mappings, f_1 and f_2, that map these tensor spaces into themselves, satisfying:

$$f_1(W_2) \subset W_2 \quad (2.12) f_2(V_2) \subset V_2 \quad (2.13)$$

For a linear mapping Ψ_x defined at x, we construct a relationship between the multi-vector space G and the image $f_1(W_2)$ such that $\Psi_x(G) \subset f_1(W_2)$. Combining this with condition (2.12) confirms that:

$$\Psi_x(G) \subset W_2 \quad (2.14)$$

Similarly, for a secondary linear mapping ϕ_y defined at y, we construct a relationship between the multi-vector space H and the image $f_2(V_2)$ such that $\phi_y(H) \subset f_2(V_2)$. Combining this with condition (2.13) yields:

$$\phi_y(H) \subset V_2 \quad (2.15)$$

By relating the mapped multi-vector space $\Psi_x(G)$ to the core connection mapping $f(W_2)$, identity (2.11) implies the tracking chain:

$$\Psi_x(G) \subset f(W_2) \subset W_2 \subset V_2 \quad (2.16)$$

We then construct the remaining bounds of the multi-vector system to generate an overarching inclusion mapping across all component spaces:

$$\Psi_x(G) \subset f(W_2) \subset W_2 \subset \phi_y(H) \subset f_2(V_2) \subset V_2 \quad (2.17)$$

From the structural dependencies built into (2.11) and the master chain (2.17), we observe that the baseline connection implies the extended configuration, and vice versa:

$$(2.11) \Longrightarrow (2.17) \quad \text{and} \quad (2.17) \Longrightarrow (2.14) \quad (2.18)$$

Therefore, by transitivity, we establish the direct implication:

$$(2.11) \Longrightarrow (2.14) \quad (2.19)$$

Following the same logical pathway for the secondary spaces in (2.15) and (2.17), we obtain:

$$(2.17) \Longrightarrow (2.15) \quad \text{and} \quad (2.11) \Longrightarrow (2.15) \quad (2.20)$$

The validity of implication (2.19) proves that there exists a valid, structurally coherent tensorial form Ψ_x corresponding to the tensor-connexion f. Concurrently, the validity of implication (2.20) proves that there exists an alternative, equally valid tensorial form ϕ_y bound to the exact same tensor-connexion f. Combining these two conclusions, we demonstrate that a single tensor-connexion f simultaneously admits two unique tensorial forms, Ψ_x and ϕ_y, over the manifold domain.

3. CONCLUSION:

In this paper, we examined the algebraic containment and mapping properties of connections on a differentiable manifold X. First, we demonstrated that the tensor product of tangent spaces evaluated at two distinct endpoints along a curve ($T_Q \otimes T_P$) can achieve structural identity with the tensor product of intermediate tangent spaces ($T_S \otimes T_R$) under specific parallel transport routing. Second, we proved that a singular tensor-connexion f operating across multi-vector and type-specific tensor hulls yields two distinct tensorial forms, Ψ_x and ϕ_y. This establishes that a single connection operator can be represented through dual geometric lenses depending on how the localized multi-vector spaces are configured, providing a broader framework for studying field equations and coordinate transformations in tensor analysis.

REFERENCES:

[1]. Bishop, R. L. and Crittenden, R. J. Geometry of Manifolds. Academic Press, New York (1964).

[2]. Bishop, R. L. and Goldberg, S. I. Tensor Analysis of Manifolds. Macmillan, New York (1968).

[3]. Eisenhart, L. P. A Treatise on the Differential Geometry of Curves and Surfaces. Ginn and Co., New York (1975).

[4]. Flanders, H. Differential Forms. Academic Press, New York (1968).

[5]. Lang, S. Introduction to Differentiable Manifolds. Interscience Publishers, New York (1962).